χ-bounded families of oriented graphs

Pierre Aboulker, Jørgen Bang-Jensen, Nicolas Bousquet, Pierre Charbit, Frédéric Havet, Frédéric Maffray, Jose Zamora

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P. Aboulker[†] J. Bang-Jensen[‡] N. Bousquet[§] P. Charbit[¶] F. Havet[†]
F. Maffray[||] J. Zamora[**]

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Abstract

A famous conjecture of Gyárfás and Sumner states for any tree T and integer k, if the chromatic number of a graph is large enough, either the graph contains a clique of size k or it contains T as an induced subgraph. We discuss some results and open problems about extensions of this conjecture to oriented graphs. We conjecture that for every oriented star S and integer k, if the chromatic number of a digraph is large enough, either the digraph contains a clique of size k or it contains S as an induced subgraph. As an evidence, we prove that for any oriented star S, every oriented graph with sufficiently large chromatic number contains either a transitive tournament of order 3 or S as an induced subdigraph. We then study for which sets $\mathcal{P}$ of orientations of P_4 (the path on four vertices) similar statements hold. We establish some positive and negative results.

1 Introduction

What can we say about the induced subgraphs of a graph G with large chromatic number? Of course, one way for a graph to have large chromatic number is to contain a large complete subgraph. However, if we consider graphs with large chromatic number and small clique number, then we can ask what other subgraphs must occur. We can avoid any graph H that contains a

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[†]Project Coati, I3S (CNRS, UNSA) and INRIA, Sophia Antipolis, France.

[‡]Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark. Part of this work was done while this author was visiting project COATI, Sophia Antipolis. Hospitality and financial support from Labex UCN@Sophia, Sophia Antipolis is gratefully acknowledged. The research of Bang-Jensen was also supported by the Danish research council under grant number 1323-00178B.

[§]LIRIS, Ecole Centrale Lyon, France.

[¶]IRIF, Université Paris Diderot, France and Project Gang , INRIA.

[||]G-SCOP, CNRS and Université Grenoble Alpes, Grenoble, France.

[**]Departamento de Matemáticas, Universidad Andres Bello, Santiago, Chile. Partially supported by Fondecyt Regular 1160975, Nucleo Milenio Información y Coordinación en Redes ICM/FIC RC130003 and Basal PFB-03 CMM, Universidad de Chile.

cycle because, as proved by Erdős [8], there are graphs with arbitrarily high girth and chromatic number; but what can we say about trees? Gyárfás [14] and Sumner [29] independently made the following beautiful and difficult conjecture.

Conjecture 1 (Gyárfás [14] and Sumner [29])**.** For every integer k and tree T, there is an integer $f(k,T)$ such that every graph with chromatic number at least $f(k,T)$ contains either a clique of size k, or an induced copy of T.

We can rephrase this conjecture, using the concept of χ-bounded graph classes introduced by Gyárfás [15]. A class of graph $\mathcal{G}$ is said to be χ-*bounded* if there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for every $G \in \mathcal{G}$; such a function f is called a χ-*bounding* function. For instance, the class of perfect graphs is χ-bounded with $f(k) = k$ as a χ-bounding function.

For a graph H, we write Forb(H) for the class of graphs that do not contain H as an induced subgraph. For a class of graphs $\mathcal{H}$, we write Forb$(\mathcal{H})$ for the class of graphs that contain no member of $\mathcal{H}$ as an induced subgraph. As we have remarked, Forb(H) is not χ-bounded when H contains a cycle. The conjecture of Gyárfás and Sumner (Conjecture 1) asserts that Forb(T) is χ-bounded for every tree T. In fact, an easy argument shows that the conjecture is equivalent to the following one .

Conjecture 2. Forb(H) is χ-bounded if and only if H is a forest.

There are not so many cases solved for this conjecture, let us recall the main ones.

- Stars: Ramsey's Theorem implies easily that Forb$(K_{1,t})$ is χ-bounded for every t.
- Paths: Gyárfás [15] showed that Forb(P) is χ-bounded for every path P.
- Trees of radius 2: using the previous result, Kierstead and Penrice [19] proved that Forb(T) is χ-bounded for every tree T of radius two (generalizing an argument of Gyárfás, Szemerédi and Tuza [16] who proved the triangle-free case). This result is proved using a result, attributed to Hajnal and Rödl (see [19]) but apparently denied by Hajnal (see [20]), stating that Forb$(\{T, K_{n,n}\})$ is χ-bounded for every tree T and every integer n.
- Subdivision of stars: it is a corollary of the following topological version of Conjecture 1 established by Scott [27]: *for every tree T and integer k, there is $g(k,T)$ such that every graph G with $\chi(G) > g(k,T)$ contains either a clique of size k or an induced copy of a subdivision of T.*

More generally, if $\mathcal{H}$ is a finite class of graphs, then Forb$(\mathcal{H})$ is χ-bounded only if $\mathcal{H}$ contains a forest, and Conjecture 2 states that the converse is true. In contrast, there are infinite classes of graphs $\mathcal{H}$ containing no trees that are χ-bounded. A trivial example is the set of odd cycles, since graphs with no (induced) odd cycles are bipartite. Another well-known example are *Berge graphs* which are the graphs with no odd holes and no odd anti-holes as induced subgraphs. A *hole* is an induced cycle of length at least 4. An *antihole* is an induced subgraph that is the complement of a hole. A hole or antihole is *odd* (resp. *even*) if it has a odd (resp. even) number of vertices. The celebrated Strong Perfect Graph Theorem [5] states that Berge graphs are perfect graphs, i.e. graphs such that each induced subgraph has its chromatic number equal to its clique number. In particular, the class of Berge graphs is χ-bounded with the identity as bounding function. Many super-classes of the class of Berge graphs are conjectured or proved to be χ-bounded. In fact, Scott and Seymour [28] proved that if G is odd-hole-free, then $\chi(G) \leqslant 2^{3^{\omega(G)}}$. This upper bound is certainly not tight. Better bounds are known for small values of $\omega(G)$. If $\omega(G) = 2$, then G has no odd cycles and so is bipartite. If $\omega(G) = 3$, then $\chi(G) \leqslant 4$ as shown by Chudnovsky et al. [6].

Theorem 3 (Chudnovsky et al. [6])**.** *Every odd-hole-free graph with clique number at most* 3 *has chromatic number at most* 4.

The goal of this paper is to extend some results known about Conjecture 1. Let T be a tree for which we know this conjecture is true, and let $\mathcal{D}_T$ be a set of orientations of T. Then one can consider the class $\text{Forb}(\mathcal{D})$ of oriented graphs that have an orientation without any induced subdigraph in $\mathcal{D}_T$. Different sets $\mathcal{D}_T$ will define different superclasses of $\text{Forb}(T)$, and one can wonder which of these are still χ-bounded. Equivalently, if one defines the chromatic number or clique number of an oriented graph to be that of its underlying graph, one can also talk about χ-bounded classes of oriented graphs, and we can ask which set of oriented trees, when forbidden as induced subdigraphs, defines χ-bounded classes of oriented graphs. After a section establishing notations and basic tools, we consider oriented stars (i.e. orientations of $K_{1,n}$) and oriented paths (i.e. orientations of paths).

Before detailing those results, let us note that in this oriented setting, if we do not ask the subdigraph to be induced, then the problem is radically different. Burr proved that every $(k-1)^2$-chromatic oriented graph contains every oriented tree of order k. This was slightly improved by Addario-Berry et al. [1] by replacing $(k-1)^2$ by $(k^2/2 - k/2 + 1)$. The right bound is conjectured [4] to be $(2k-2)$.

1.1 Oriented stars

We conjecture the following :

Conjecture 4. For any oriented star S, $\text{Forb}(S)$ is χ-bounded.

For any positive integers k, ℓ, we denote by $S_{k,\ell}$ the oriented star on $k+\ell+1$ vertices where the center has in-degree k and out-degree ℓ. Of course, by directional duality, the result for $S_{k,\ell}$ implies the result for $S_{\ell,k}$. Also, proving the conjecture for $S_{k,k}$ for all values of k is enough since $\text{Forb}(S_{k,\ell}) \subseteq \text{Forb}(S_{k,k})$ if $k \geqslant \ell$.

The cases $k = 0$ and $k = \ell = 1$ are not difficult and were previously known (as mentioned in [20]) but no proof was published. As those proofs are short and interesting, we provide them in Subsections 3.1.

By definition of χ-boundedness, Conjecture 4 can be restated as follows: for every positive integer p, $\text{Forb}(\text{Or}(K_p), S)$ has bounded chromatic number, where $\text{Or}(K_p)$ is the set of orientations of K_p. There are exactly two orientations of K_3 : the directed cycle on three vertices $\vec{C}_3$, and the transitive tournament on three vertices TT_3. It is not difficult to show that, for any oriented star S, Forb$(\vec{C}_3, TT_3, S)$ has bounded chromatic number. We can even determine the exact value of $\chi(\text{Forb}(\vec{C}_3, TT_3, S))$ (Proposition 14). This can be seen as the first step ($p = 3$) of Conjecture 4. Kierstead and Rödl [20] proved that Forb$(\vec{C}_3, S)$ is χ-bounded. In Theorem 15, we prove the following counterpart : Forb(TT_3, S) *has bounded chromatic number, for all oriented star* S. This can be seen as the next step towards Conjecture 4; indeed, by Theorem 6, every orientation of K_4 contains TT_3 as an induced subdigraph, so Forb$(TT_3, S) \subset$ Forb$(\text{Or}(K_4), S)$. The next step would be to prove that Forb$(\text{Or}(K_4), S)$ has bounded chromatic number for every star S.

1.2 Oriented paths on four vertices

Let us denote by P_k the path on k vertices. Since P_2 and P_3 are stars, the next case for paths concerns orientations of P_4. The graphs with no induced P_4 are known as *cographs*, and it is well-known that cographs are perfect. In particular, the class of cographs is χ-bounded (or equivalently Forb$(\text{Or}(P_4))$ is χ-bounded). There are four non-isomorphic orientations of P_4. They are depicted in Figure 1.

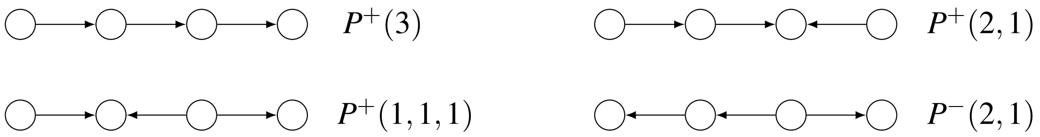

Figure 1: The four orientations of P_4

In Section 4, we study Forb$(\mathcal{P})$ when $\mathcal{P}$ is a set of orientations of P_4. Kierstead and Trotter [21] proved that Forb$(P^+(3))$ is not χ-bounded by constructing $(TT_3, P^+(3))$-free oriented graphs with arbitrary large chromatic number. Gyárfás pointed out that the natural orientations of the so-called *shift graphs* ([9]) are in Forb$(\vec{C}_3, TT_3, P^+(1,1,1))$ but may have arbitrarily large chromatic number. Consequently, Forb$(P^+(1,1,1))$ is not χ-bounded. See Subsection 4.1.

We believe that $\{P^+(3)\}$ and $\{P^+(1,1,1)\}$ are the only non-empty subsets $\mathcal{P}$ of Or(P_4) such that Forb$(\mathcal{P})$ is not χ-bounded.

Conjecture 5. Let $\mathcal{P}$ be a non-empty subset of Or(P_4). If $\mathcal{P} \neq \{P^+(3)\}$ and $\mathcal{P} \neq \{P^+(1,1,1)\}$, then Forb$(\mathcal{P})$ is χ-bounded.

We prove this conjecture in the case when $P^+(3) \in \mathcal{P}$: in Corollary 35, we show that the classes Forb$(P^+(3), P^+(2,1))$, Forb$(P^+(3), P^-(2,1))$, and Forb$(P^+(3), P^+(1,1,1))$ are χ-bounded. Hence, it remains to prove Conjecture 5 for $\mathcal{P} \subseteq$ Forb$(P^+(2,1), P^-(2,1), P^+(1,1,1))$. Several results in this direction have been established. Kierstead (see [26]) proved that every $(\vec{C}_3, P^+(2,1), P^-(2,1))$-free oriented graph D can be coloured with $2^{\omega(D)} - 1$ colours, in particular Forb$(\vec{C}_3, P^+(2,1), P^-(2,1))$ is χ-bounded. Chvátal [7] proved that acyclic $P^+(2,1)$-free oriented graphs are perfect, so Forb$(\vec{\mathcal{C}}, P^+(2,1))$ is χ-bounded. Kierstead and Rödl [20] generalized those two results by proving (but with a larger bounding function) that Forb$(\vec{C}_3, P^+(2,1))$

is χ-bounded. In Subsection 4.2, we make the first two steps towards the χ-boundedness of Forb$(P^+(2,1))$. We prove $\chi(\text{Forb}(\vec{C}_3, TT_3, P^+(2,1)) = 3$ and $\chi(\text{Forb}(TT_3, P^+(2,1)) = 4$.

2 Definitions, notations and useful facts

Let D be a digraph. If uv is an arc, we say that u *dominates* v and write $u \to v$. Let $P = (x_1, x_2, \dots, x_n)$ be an oriented path. We say that P is an (x_1, x_n)*-path*. The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. Then we say that P is a *directed path* or simply a *dipath*, if $x_i \to x_{i+1}$ for all $1 \leqslant i \leqslant n-1$. An oriented cycle $C = (x_1, x_2, \dots, x_n, x_1)$ is a *directed cycle*, if $x_i \to x_{i+1}$ for all $1 \leqslant i \leqslant n$, where $x_{n+1} = x_1$. The directed cycle of length k is denoted by $\vec{C}_k$.

The digraph D is *connected* if its underlying graph is connected. It is *strongly connected*, or *strong*, if for any two vertices u, v, there is a (u, v)-dipath in D. A strong component U is *initial* if all the arcs with head in U have their tail in U. We denote by $\vec{\mathcal{C}}$ the class of directed cycles and by $\mathcal{S}$ the class of strong oriented graphs.

The *chromatic number* (resp. *clique number*) of a digraph, denoted by $\chi(D)$ (resp. $\omega(D)$), is the chromatic number (resp. clique number) of its underlying graph. The *chromatic number* of a class $\mathcal{D}$ of digraphs, denoted by $\chi(\mathcal{D})$, is the smallest k such that $\chi(D) \leqslant k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such k exists. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ *has bounded chromatic number*. Similarly to undirected graphs, a class of oriented graphs $\mathcal{D}$ is said to be χ*-bounded* if there is a function f such that $\chi(D) \leqslant f(\omega(D))$ for every $D \in \mathcal{D}$. Such a function f is called a χ*-bounding function* for the class.

Let F be a digraph and let $\mathcal{F}$ be a class of digraphs. A digraph is F*-free* (resp. $\mathcal{F}$*-free*) if it does not contain F (resp. any element of $\mathcal{F}$) as an induced subgraph. In this paper, we study for which classes $\mathcal{F}$ of digraphs, the class of $\mathcal{F}$-free digraphs is χ-bounded. Observe that such an $\mathcal{F}$ must contain a complete (symmetric) digraph, that is a digraph in which any two distinct vertices are joined by two arcs in opposite direction. Indeed, $\vec{K}_k$, the complete digraph on k vertices, has chromatic number k, and every induced subdigraphs of a complete digraph is a complete digraph.

In this paper, **we consider oriented graphs**, which are $\vec{K}_2$-free digraphs. Alternately, an *oriented graph* may be defined as the orientation of a graph. Note that an $\mathcal{F}$-free oriented graph is an $(\mathcal{F}, \vec{K}_2)$-free digraph. We denote by Forb$(\mathcal{F})$ the class of $\mathcal{F}$-free oriented graphs. We are interested in determining for which class $\mathcal{F}$ of oriented graphs, the class Forb$(\mathcal{F})$ is χ-bounded. To keep notation simple, we abbreviate Forb$(\{F_1, \dots, F_p\})$ in Forb$(F_1, \dots, F_p)$, Forb$(\mathcal{F}_1 \cup \dots \cup \mathcal{F}_p)$ in Forb$(\mathcal{F}_1, \dots, \mathcal{F}_p)$, Forb$(\{F\} \cup \mathcal{F}\})$ in Forb$(F, \mathcal{F})$, and so on ...

Let us denote by Or(G) the set of all possible orientations of a graph G, and by Or$(\mathcal{G})$ the set of all possible orientations of a graph in the class $\mathcal{G}$. By definition a class of oriented graphs $\mathcal{D}$ is χ-bounded if and only if for all positive integer n, $\mathcal{D} \cap$ Forb(Or(K_n)) has bounded chromatic number. The Gyárfás-Sumner conjecture (Conjecture 2) can be restated as follows : Forb(Or(H)) is χ-bounded if and only if H is a forest. However Forb$(\mathcal{F})$ could be χ-bounded for some strict subset $\mathcal{F}$ of Or(H). More generally, for any result proving that a class Forb$(\mathcal{G})$ is χ-bounded, a natural question is to ask for which subsets $\mathcal{F}$ of Or$(\mathcal{G})$, the class Forb$(\mathcal{F})$ is also χ-bounded. For example, the result mentioned in the introduction stating that Forb$(\{T, K_{n,n}\})$

is χ-bounded for every tree T and every integer n has been generalized to orientations of $K_{n,n}$ and oriented trees : Kierstead and Rödl [20] proved that for any positive integer n and oriented tree T, the class $\mathrm{Forb}(DK_{n,n}, T)$ is χ-bounded where $DK_{n,n}$ is the orientation of the complete bipartite graph $K_{n,n}$ where all edges are oriented from a part to the other.

A *tournament* is an orientation of a complete graph. The unique orientation of K_n (the complete graph on n vertices) with no directed cycles is called the *transitive tournament* of order n and is denoted TT_n. Let $\mathrm{tt}(D)$ be the order of a largest transitive tournament in D. Observe that a class of oriented graphs is χ-bounded if and only if there is a function g such that $\chi(D) \leqslant g(\mathrm{tt}(D))$ for every $D \in \mathcal{D}$ thanks to the following result due to Erdős and Moser [10].

Theorem 6 (Erdős and Moser [10]). *For every tournament T,* $\mathrm{tt}(T) \geqslant 1 + \lfloor \log |V(T)| \rfloor$.

By the above observation, $\mathcal{D}$ is χ-bounded if and only if for all positive integer n, $\mathcal{D} \cap \mathrm{Forb}(TT_n)$ has bounded chromatic number. Also let us remark that any orientation of K_4 contains a TT_3, so $\mathcal{D} \cap \mathrm{Forb}(\mathrm{Or}(K_3)) \subset \mathcal{D} \cap \mathrm{Forb}(TT_3) \subset \mathcal{D} \cap \mathrm{Forb}(\mathrm{Or}(K_4))$. When we are able to prove the result for $\mathrm{Or}(K_3)$-free oriented graphs and want to extend the result to $\mathrm{Or}(K_4)$-free ones, an intermediate step is therefore to prove the TT_3-free case. We will do this in some cases in Section 3 and 4.

For a set or subgraph S of D, we denote by $\mathrm{Reach}^+_D(S)$ (resp. $\mathrm{Reach}^-_D(S)$), the set of vertices x such that there is a directed out-path (resp. directed in-path) with initial vertex in S and terminal vertex x.

Let D be a digraph on n vertices $v_1, \dots, v_n$. A digraph D' is an *extension* of D if $V(D')$ can be partitioned into $(V_1, \dots, V_n)$ such that $A(D') = \{xy \mid x \in V_i, y \in V_j \text{ and } v_iv_j \in A(D)\}$. Observe that some V_i may be empty. In particular, induced subdigraphs of D are extension of D.

To finish this section let us state easy results that we will often use in the proofs. Recall that a k-critical graph is a graph of chromatic number k of which any strict subgraph has chromatic number at most $k-1$. For a digraph D, $\delta(D)$ denotes the minimum degree of the underlying non-oriented graph, and $\Delta^+(D)$ (resp. $\Delta^-(D)$) denote the maximum out-degree (resp. in-degree) of D.

Proposition 7. *If D is a k-critical digraph, then D is connected, $\delta(D) \geqslant k-1$ and $\Delta^+(D), \Delta^-(D) \geqslant (k-1)/2$.*

Theorem 8 (Brooks [3]). *Let G be a connected graph. If G is not a complete graph or an odd cycle, then $\chi(G) \leqslant \Delta(G)$.*

3 Forbidding Oriented Stars

In this section, we study the χ-boundedness of $\mathrm{Forb}(S)$, for S an oriented star.

3.1 Forbidding $S_{0,\ell}$ or $S_{1,1}$

In [20], the authors state that the results in this section were already known, but since they give no reference, and the proofs are short, we include them here. As written in the introduction, the fact that $\mathrm{Forb}(K_{1,t})$ is χ-bounded follows directly from the following celebrated theorem due to Ramsey.

Theorem 9 (Ramsey [24])**.** *Given any positive integers s and t, there exists a smallest integer $\mathbf{r}(s,t)$ such that every graph on at least $\mathbf{r}(s,t)$ vertices contains either a clique of s vertices or a stable set of t vertices.*

Similarly, it can be used to show that $\mathrm{Forb}(S_{0,\ell})$ is χ-bounded.

Theorem 10. *Let ℓ be a positive integer. If $D \in \mathrm{Forb}(S_{0,\ell})$, then $\chi(D) < 2\mathbf{r}(\omega(D),\ell)$.*

Proof. Let D be an $S_{0,\ell}$-free oriented graph. If $\chi(D) \geqslant 2\mathbf{r}(s,\ell)$, then by Proposition 7, D has a vertex v with in-degree at least $\mathbf{r}(\omega,\ell)$. Now $N^-(v)$ contains no stable set of size ℓ, for its union with v would induce an $S_{0,\ell}$. Therefore, by Theorem 9, $N^-(v)$ contains a clique of s vertices, which forms a clique of size $s+1$ with v. □

Note that using Ramsey's Theorem is the only known way to prove that $\mathrm{Forb}(K_{1,t})$ and $\mathrm{Forb}(S_{0,\ell})$ are χ-bounded. The resulting bounding functions are very high and certainly very far from being tight.

Proposition 11. *(i) $\chi(\mathrm{Forb}(TT_3,S_{0,2})) = 3$.*
(ii) For $\ell \geqslant 2$, $\chi(\mathrm{Forb}(TT_3,S_{0,\ell})) \leqslant 2\ell-2$.

Proof. (i) A digraph in $\mathrm{Forb}(TT_3,S_{0,2})$ has no vertex of out-degree at least 2. Hence by Proposition 7, it contains no 4-critical digraph. Thus $\chi(\mathrm{Forb}(TT_3,S_{0,2})) \leqslant 3$.

The directed odd cycles are in $\mathrm{Forb}(TT_3,S_{0,2})$ and have chromatic number 3. This implies that $\chi(\mathrm{Forb}(TT_3,S_{0,2})) = 3$.

(ii) It suffices to prove that every critical digraph D in $\mathrm{Forb}(TT_3,S_{0,\ell})$ has chromatic number at most $2\ell-2$. Observe that for every vertex v, $N^+(v)$ induces a stable set because D is TT_3-free. Thus $d^+(v) \leqslant \ell-1$ since D is $S_{0,\ell}$-free. Hence $|A(D)| \leqslant (\ell-1)|V(D)|$.

If D contains a vertex of degree less than $2\ell-2$, then by Proposition 7, $\chi(D) \leqslant 2\ell$. If not, then every vertex has degree exactly $2\ell-2$. Moreover, D is not a tournament of order $2\ell-1$, because every such tournament contains a TT_3. Hence by Brooks' Theorem (Theorem 8), $\chi(D) \leqslant 2\ell-2$. □

$S_{1,1}$-free orientations of graphs are known as quasi-transitive oriented graphs, and it is a result of Ghouila-Houri ([13]) that a graph has a quasi-transitive orientation if and only if it has a transitive orientation, that is an orientation both acyclic and quasi-transitive (such graphs are commonly called *comparability graphs*). Note that if a graph has a transitive orientation, then cliques correspond to directed paths; according to a classical theorem, due independently to Gallai [12], Hasse [17], Roy [25], and Vitaver [30], the chromatic number of a digraph is at most the number of vertices of a directed path of maximum length : this implies that comparability graphs are perfect. Hence $S_{1,1}$-free oriented graphs are oriented perfect graphs, and therefore χ-bounded.

Oriented graphs in $\mathrm{Forb}(\vec{C}_3,TT_3,S_{1,1})$ and in $\mathrm{Forb}(TT_3,S_{1,1})$ actually have a very simple structure as we show now.

Theorem 12. *Every connected $(TT_3, S_{1,1})$-free oriented graph D satisfies the following:*

(i) If D is $\vec{C}_3$-free, then D is an extension of TT_2.

(ii) D is an extension of $\vec{C}_3$.

Proof. All vertices of an oriented graph $D \in \text{Forb}(\vec{C}_3, TT_3, S_{1,1})$ are clearly either source or sink, which implies (i).

Now let $D \in \text{Forb}(TT_3, S_{1,1})$. If D does contain no $\vec{C}_3$, then it is an extension of TT_2 (and thus of $\vec{C}_3$) and we are done. So we may assume that D contains a $\vec{C}_3$. Let (A,B,C) be the partition of a maximal extension of $\vec{C}_3$ in D such that none of the sets A, B, C is empty, where all the arcs are from A to B, from B to C and from C to A. If $A \cup B \cup C = V(D)$ we are done, so we may assume without loss of generality that there exists adjacent vertices $a \in A$ and $x \in D - (A \cup B \cup C)$. By directional duality we may assume that $a \to x$. For all $c \in C$, c is adjacent to x for otherwise $\{c,a,x\}$ induces $S_{1,1}$ and $x \to c$ for otherwise $\{a,x,c\}$ induces a TT_3.

Let $c \in C$. For all $a' \in A$, a' and x are adjacent for otherwise $\{x,c,a'\}$ induces a $S_{1,1}$, and $a' \to x$ for otherwise $\{a',x,c\}$ induces a TT_3. Moreover x is not adjacent to any vertex $b \in B$, otherwise $\{a,x,b\}$ induces a TT_3. Hence $D\langle A \cup B \cup C \cup \{x\}\rangle$ is an extension of $\vec{C}_3$ with partition $(A,B,C \cup \{x\})$, a contradiction to the maximality of (A,B,C). □

Corollary 13. $\chi(\text{Forb}(\vec{C}_3, TT_3, S_{1,1})) = 2$ *and* $\chi(\text{Forb}(TT_3, S_{1,1})) = 3$.

3.2 Forbidding TT_3 and an oriented star

The triangle-free case for stars is easy.

Proposition 14. *Let k and ℓ be two positive integers.* $\chi(\text{Forb}(\vec{C}_3, TT_3, S_{k,\ell})) \leqslant 2k + 2\ell - 2$.

Proof. Let D be a $(2k+2\ell-1)$-critical $(\vec{C}_3, TT_3)$-free oriented graph. Let V^- be the set of vertices of in-degree less than k and let V^+ be the set of vertices of out-degree less than ℓ. By Proposition 7, $\chi(D\langle V^-\rangle) \leqslant 2k-1$ and $\chi(D\langle V^+\rangle) \leqslant 2\ell-1$. Consequently, $V^- \cup V^+ \neq V(D)$ for otherwise D would be $(2k+2\ell-2)$-colourable. Hence, there is a vertex v with in-degree at least k and out-degree at least ℓ. Thus v is the center of an $S_{k\ell}$, which is necessarily induced because D is $(\vec{C}_3, TT_3)$-free. □

Kierstead and Rödl [20] proved that the class $\text{Forb}(\vec{C}_3, S_{k,\ell})$ is χ-bounded (without providing any explicit bound). The goal of this section is to prove the following counterpart to that theorem.

Theorem 15. *For every positive integers k and ℓ, the class* $\text{Forb}(TT_3, S_{k,\ell})$ *has bounded chromatic number.*

As mentioned in the introduction this can be seen as the next step towards Conjecture 4 because $\text{Forb}(\text{Or}(K_3), S_{k,\ell})) \subset \text{Forb}(TT_3, S_{k,\ell})) \subset \text{Forb}(\text{Or}(K_4), S_{k,\ell}))$.

As noted before, in order to prove Theorem 15 it suffices to prove the following one.

Theorem 16. *For every positive integer k,* $\text{Forb}(TT_3, S_{k,k})$ *has bounded chromatic number.*

The proof of Theorem 16 is given in the next subsections.

3.2.1 Reducing to triangle-free colouring

Let D be a digraph. A *triangle-free colouring* is a colouring of the vertices such that no triangle is monochromatic. The *triangle-free chromatic number, denoted by* $\chi_{\mathrm{T}}(D)$, of D is the minimum number of colours in a triangle-free colouring of D.

Lemma 17. *If* $D \in \mathrm{Forb}(TT_3, S_{k,k})$*, then* $\chi(D) \leqslant (4k-2) \cdot \chi_{\mathrm{T}}(D)$

Proof. Let $V_1, \dots, V_{\chi_{\mathrm{T}}(D)}$ be a triangle-free colouring of D. For every $1 \leqslant i \leqslant \chi_{\mathrm{T}}(D)$, the graph $D[V_i]$ is $(\vec{C}_3, TT_3, S_{k,k})$-free and thus $\chi(D[V_i]) \leqslant 4k-2$ by Proposition 14. It follows that $\chi(D) \leqslant (4k-2) \cdot \chi_{\mathrm{T}}(D)$. □

Lemma 17 implies that in order to prove Theorem 16 it is sufficient to prove the following theorem.

Theorem 18. *For any positive integer k,* $\chi_{\mathrm{T}}(\mathrm{Forb}(TT_3, S_{k,k})) < +\infty$.

We prove Theorem 18 in Section 3.2.3, and this will establish Theorem 16 as well. The proof requires several preliminaries. To make the proof clear and avoid tedious calculations, we do not make any attempt to get an explicit constant C_k such that $\chi_{\mathrm{T}}(\mathrm{Forb}(TT_3, S_{k,k})) < C_k$, because our method yields a huge constant which is certainly a lot larger than $\chi_{\mathrm{T}}(\mathrm{Forb}(TT_3, S_{k,k}))$.

3.2.2 Preliminaries

A combinatorial lemma. We start with a combinatorial lemma that will only be used to prove Lemma 21.

Lemma 19. *Let* $k \in \mathbb{N}$ *and* $p \in]0,1[$*. Then there is an integer* $N(k,p)$ *that satisfies the following:*
If $H = (V,E)$ *is any hypergraph where all hyperedges have size at least* $p|V|$*, and the intersection of any k hyperedges has size at most* $k-1$*, and* $|V| \geqslant N(k,p)$*, then* $|E| < k/p^k$*.*

Proof. Set $n = |V|$. We need to prove that if n is sufficiently large, then $|E| < k/p^k$. Let $\varphi : V^k \to \mathbb{N}$ be the function defined as follows: for any k-subset T of V, let $\varphi(T) = |\{A \in E \mid T \subseteq A\}|$. Set $\Phi = \Sigma_{T \in V^k} \varphi(T)$. By the hypothesis we have $\varphi(T) \leqslant k-1$ for all $T \in V^k$, and thus $\Phi \leqslant \binom{n}{k} \cdot (k-1)$. Since each hyperedge contributes to at least $\binom{pn}{k}$ to Φ, we have $\Phi \geqslant |E| \cdot \binom{pn}{k}$. So $|E| \cdot \binom{pn}{k} \leqslant (k-1) \cdot \binom{n}{k}$, and thus

$$|E| \leqslant (k-1) \cdot \frac{\binom{n}{k}}{\binom{pn}{k}} \sim_{n \to \infty} \frac{k-1}{p^k},$$

which implies the result. □

The constants All along the proofs we will use several constants; we describe all of them here.

- $k \geqslant 2$ is a fixed integer (that corresponds to the forbidden $S_{k,k}$).
- $s = 1 - 1/2k$.
- We choose $\varepsilon \in]0, \frac{1}{2k}[$.
- We choose $t \in]s, 1-\varepsilon[$ (we need $t > s$ in Lemma 22 and 23 and we need $t < 1-\varepsilon$ in Lemma 24).
- $g = k/(1-t-\varepsilon)^k$ (this corresponds to the constant k/p^k in Lemma 19 for $p = 1-t-\varepsilon$).
- $N_1 = \max\left(N(k, 1-t-\varepsilon), \frac{(1-t-\varepsilon)\cdot g}{\varepsilon} + g\right)$ where N is the function defined in Lemma 19.
- $N_2 = \max(N_1, \frac{g}{t-s} + g + 1)$.
- $d = \max\left(\frac{N_2}{t} + 8g, \frac{2tg}{t-s} + g\right)$.

Definitions Let D be an oriented graph and A and B be two disjoint stable sets. The graph $D[A,B]$ is the bipartite graph with parts A and B. If $D[A,B]$ is $\overline{K}_{k,k}$-free and all its arcs are from A to B, we write $A \rightsquigarrow B$. Note that $A \rightsquigarrow B$ implies $A \rightsquigarrow C$ for every $C \subseteq B$. Let $0 < \tau < 1$. By $A \rightarrow_\tau B$, we mean:

- there is no arc from B to A,
- for every $a \in A$, we have $d_B^+(a) \geqslant \tau|B|$ and
- for every $b \in B$, we have $d_A^-(b) \geqslant \tau|A|$.

If $A \rightsquigarrow B$ and $A \rightarrow_\tau B$, we write $A \rightsquigarrow_\tau B$.

The tools We now prove several lemmas that will be used in the proof.

Lemma 20. *Let $D \in \mathrm{Forb}(TT_3, S_{k,k})$. Let $x \in V(D)$. Then $N^+(x) \rightsquigarrow N^-(x)$.*

Proof. Since D is TT_3-free, $N^+(x)$ and $N^-(x)$ are stable sets and any arc between $N^+(x)$ and $N^-(x)$ has its tail in $N^+(x)$ and its head in $N^-(x)$. Since D is $S_{k,k}$-free, $D[N^-(x), N^+(x)]$ is $\overline{K}_{k,k}$-free. □

The next lemma roughly states that if A and B are two large enough disjoint stable sets such that $A \rightsquigarrow B$, then up to deleting a few vertices from A and B we have $A \rightsquigarrow_t B$.

Lemma 21. *Let A, B be two disjoint stable sets such that $A \rightsquigarrow B$. If $|A|, |B| \geqslant N_1$, then there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ such that:*

- $|A_1| \geqslant |A| - g$, $|B_1| \geqslant |B| - g$ *and*

- $A_1 \rightsquigarrow_t B_1$.

Proof. Assume that $|A|, |B| \geqslant N_1$. Let

$$A_2 = \{a \in A : d_B^+(a) < (t+\varepsilon)|B|\} \text{ and } A_1 = A \setminus A_2, \text{ and}$$
$$B_2 = \{b \in B : d_A^-(b) < (t+\varepsilon)|A|\} \text{ and } B_1 = B \setminus B_2.$$

Let us first prove that both $|A_2|$ and $|B_2|$ are at most g. Consider the hypergraph $H_B = (B, E_B)$ where $E_B = \{B \setminus N^+(a) \mid a \in A_2\}$. We have $|E_B| = |A_2|$ and the size of each hyperedge of H_B is at least $(1-t-\varepsilon)|B|$. Since $D[A,B]$ is $\overline{K}_{k,k}$-free, k vertices of A_2 cannot have k common non-neighbours, i.e., the intersection of any k hyperedges of H_B is at most $(k-1)$. Since $|B| \geqslant N_1 \geqslant N(k, 1-t-\varepsilon)$, Lemma 19 ensures that $|A_2| = |E_B| \leqslant \frac{k}{(1-t-\varepsilon)^k} = g$. Thus $|A_1| \geqslant |A| - g$. Similarly $|B_2| \leqslant g$ and so $|B_1| \geqslant |B| - g$.

Since $A \rightsquigarrow B$, we have $A_1 \rightsquigarrow B_1$. Thus it remains to prove that $A_1 \rightarrow_t B_1$. Since $d_B^+(a) \geqslant (t+\varepsilon)|B|$ for every $a \in A_1$, we have:

$$\begin{aligned} d_{B_1}^+(a) &\geqslant (t+\varepsilon)|B| - |B_2| \\ &\geqslant t \cdot |B_1| + \varepsilon|B_1| - (1-t-\varepsilon)|B_2|) \qquad \text{(because } |B| = |B_1| + |B_2|) \end{aligned}$$

Now $|B_2| \leqslant g$ and by definition of N_1, we have: $|B_1| \geqslant |B| - g \geqslant N_1 - g \geqslant \frac{(1-t-\varepsilon)\cdot g}{\varepsilon}$. So $\varepsilon|B_1| \geqslant (1-t-\varepsilon)|B_2|$. Consequently, $d_{B_1}^+(a) \geqslant t \cdot |B_1|$.

Similarly, we obtain $d_{A_1}^-(b) \geqslant t \cdot |A_1|$ for all $b \in B_1$, which completes the proof. □

Lemma 22. *Let $\tau \in]s,1[$ and $D \in \mathrm{Forb}(TT_3, S_{k,k})$. Let A,B,C be three disjoint stable sets of D. If for every $a \in A$, $d_B^+(a) \geqslant \tau|B|$ and for every $c \in C$, $d_B^-(c) \geqslant \tau|B|$, then $C \rightsquigarrow A$.*

Proof. Let us first prove that there is no arc from A to C. Let $a \in A$ and $c \in C$. Since $p > \frac{1}{2}$, we have $d_B^+(a) \geqslant \tau|B| > \frac{1}{2}|B|$ and $d_B^-(c) \geqslant \tau|B| > \frac{1}{2}|B|$. So there exists $b \in B$ such that $b \in N^+(a) \cap N^-(c)$, hence ac is not an arc otherwise $\{a,b,c\}$ would induce a TT_3, a contradiction.

It remains to prove that $D[C,A]$ is $\overline{K}_{k,k}$-free. Assume for contradiction that there exist $A_k = \{a_1, \dots, a_k\} \subseteq A$ and $C_k = \{c_1, \dots, c_k\} \subseteq C$ such that there is no arc between A_k and C_k. For each $a_i \in A_k$, at most $(1-\tau)|B|$ vertices in B are not in $N^+(a_i)$. Similarly for each $c_i \in C_k$, at most $(1-\tau)|B|$ vertices in B are not in $N^-(c_i)$. Thus the size of $X := \bigcap_{1 \leqslant i \leqslant k} N^+(a_i) \cap \bigcap_{1 \leqslant i \leqslant k} N^-(c_i)$ is at least $(1 - 2k(1-\tau))|B| > (1-2k(1-s))|B| = 0$. Since X is non-empty, it contains a vertex x. The set $\{x\} \cup A_k \cup C_k$ induces $S_{k,k}$, a contradiction. □

All along the paper, we often use this lemma with stronger assumptions.

Corollary 23. *Let $D \in \mathrm{Forb}(TT_3, S_{k,k})$ and $\tau \in]s,1[$. Let A,B,C be three disjoint stable sets of D. If $A \rightarrow_\tau B \rightarrow_\tau C$, then $C \rightsquigarrow A$.*

The next lemma roughly ensures that if A,B,C are three large enough stable sets such that $A \rightsquigarrow B \rightsquigarrow C$, then, up to deleting a few vertices from A and C, we have $C \rightsquigarrow A$.

Lemma 24. *Let $D \in \text{Forb}(TT_3, S_{k,k})$ and let A, B, C be three disjoint stable sets of D. If $A \rightsquigarrow B \rightsquigarrow C$ and $|A|, |B|, |C| \geqslant N_2$, then there exist $A_1 \subseteq A$, and $C_1 \subseteq C$ such that:*

- $|A_1| \geqslant |A| - g$, $|C_1| \geqslant |C| - g$ *and*
- $C_1 \rightsquigarrow A_1$

Proof. The proof consists in combining Lemmas 21 and 22. Since $A \rightsquigarrow B$ and $|A|, |B| \geqslant N_1$, Lemma 21 ensures that there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $|A_1| \geqslant |A| - g$, $|B_1| \geqslant |B| - g$ and $A_1 \rightsquigarrow_t B_1$. Similarly, since $B \rightsquigarrow C$ and $|B|, |C| > N_1$, there exist $B_2 \subseteq B$ and $C_1 \subseteq C$ such that $|B_2| \geqslant |B| - g$, $|C_1| \geqslant |C| - g$ and $B_2 \rightsquigarrow_t C_1$.
Set $B_3 = B_1 \cap B_2$ and observe that $|B_3| \geqslant |B| - 2g$. Note moreover that both $B_1 \setminus B_3$ and $B_2 \setminus B_3$ have size at most g. For all $a \in A_1$, since $A_1 \rightarrow_t B_1$ and $B_2 \rightarrow_t C_1$, we have:

$$d^+_{B_3}(a) \geqslant t|B_1| - g \geqslant \left(t - \frac{g}{|B_1|}\right)|B_3| \quad \text{and} \quad d^-_{B_3}(c) \geqslant t|B_2| - g \geqslant \left(t - \frac{g}{|B_2|}\right)|B_3|$$

By Lemma 22, it is sufficient to prove that $t - g/|B_i| > s$ for $i = 1, 2$, which is satisfied because $|B_i| \geqslant |B| - g \geqslant N_2 - g > \frac{g}{t-s}$. □

A digraph D is *c-triangle-free-critical* if $\chi_T(D) = c$ and for all $x \in V(D)$, $\chi_T(D - \{x\}) < c$.

Lemma 25. *Let $D \in \text{Forb}(TT_3)$ be a c-triangle-free-critical digraph. Then for all $x \in V(D)$, $d^+(x) \geqslant c - 1$ and $d^-(x) \geqslant c - 1$.*

Proof. Let $x \in V(D)$. Let π be a triangle-free colouring of $D - x$ using $c - 1$ colours. Since $\chi_T(D) = c$, we cannot extend π to D using a colour in $\{1, \dots, c-1\}$. Let $i \leqslant c - 1$. Since x cannot be coloured with i, the vertex x is adjacent to two vertices u_i and v_i coloured i and such that (u_i, v_i) is an arc. Since D is TT_3-free, necessarily, $v_i \rightarrow x$ and $x \rightarrow u_i$. Now all the u_i (resp. v_i) are distinct because they are coloured differently, so $d^+(x) \geqslant c - 1$ and $d^-(x) \geqslant c - 1$. □

3.2.3 Proof of Theorem 18

We are now able to prove Theorem 18. In fact, we prove the following theorem.

Theorem 26. $\chi_T(\text{Forb}(TT_3, S_{k,k})) \leqslant 2d$.

Proof. We consider a minimal counter-example, that is, a digraph $D \in \text{Forb}(TT_3, S_{k,k})$ which is $(2d+1)$-triangle-free-critical. By Lemma 25, every vertex of D has in- and out-degree at least $2d$.

Let x be a vertex. By Lemma 20, we have $N^+(x) \rightsquigarrow N^-(x)$. Since $2d \geqslant N_1$, Lemma 21 ensures that there exists a $U \subseteq N^-(x)$ and $V \subseteq N^+(x)$ of size at least $2d - g$ such that $U \rightsquigarrow_t V$.

Let A, B be two disjoint stable sets, each of size at least $\frac{N_2}{t}$, such that $A \rightsquigarrow_t B$ and maximizing $|A| + |B|$. We have:

$$|A| + |B| \geqslant 4d - 2g. \tag{1}$$

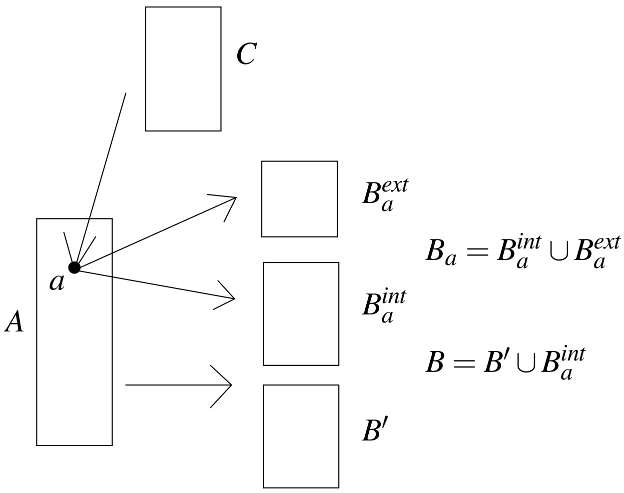

Figure 2: The situation in Case 1.

Claim 26.1. *There exists $x \in A \cup B$ such that $d^+_{D-(A\cup B)}(x) \geqslant d$ and $d^-_{D-(A\cup B)}(x) \geqslant d$.*

Proof. Assume that no vertex x in $A \cup B$ satisfies $d^+_{D-(A\cup B)}(x) \geqslant d$ and $d^-_{D-(A\cup B)}(x) \geqslant d$. Let π be a triangle-free $2d$-colouring of $D-(A\cup B)$, which exists by minimality of D. For every $a \in A$, a colour c_a in $\{1,\dots,d\}$ does not appear in $\pi(N^+(a))$ or in $\pi(N^-(a))$. Similarly, for every $b \in B$, a colour c_b in $\{d+1,\dots,2d\}$ does not appear in $\pi(N^+(b))$ or in $\pi(N^-(b))$. Let π' be the colouring of D where π' agrees with π on $D-(A\cup B)$, and $\pi'(a) = c_a$ for every $a \in A$ and $\pi'(b) = c_b$ for every $b \in B$. Since D is $(2d+1)$-triangle-free-critical, there is a monochromatic oriented triangle xyz. As π is a triangle-free colouring, at least one vertex of the triangle, say x, is in $A \cup B$. By directional duality, we may assume that $x \in A$. Since the colours used to colour vertices of A and the colours used to colour vertices of B are disjoint, y and z are not in B. Moreover, since A is a stable set, y and z are in $D-(A\cup B)$. Thus there is an in-neighbour and an out-neighbour of x coloured with c_x, a contradiction with the definition of c_x. $\Diamond$

We distinguish three cases.

Case 1: $|A| \geqslant \frac{N_2}{t} + 4g$ and there exists $a \in A$ such that $d^+_{D-(A\cup B)}(a) \geqslant d$ and $d^-_{D-(A\cup B)}(a) \geqslant d$.

Set $C = N^-(a)$, $B_a = N^+(a)$, $B_a^{int} = N^+(a) \cap B$, $B_a^{ext} = B_a \setminus B_a^{int}$ and $B' = B \setminus B_a^{int}$ (see Figure 2 for a rough picture of the situation). Note that by assumption the sizes of B_a^{ext} and C are both at least d. Since $d \geqslant N_2 + 4g$, both C and B_a^{ext} have size at least $N_2 + 4g$. Because $A \leadsto_t B$ and B has size at least $\frac{N_2}{t}$, we also have

$$|B_a^{int}| \geqslant N_2.$$

Since $A \leadsto B$, we have $A \leadsto B_a^{int}$. Moreover, $B_a^{int} \subseteq N^+(a)$ and $C \subseteq N^-(a)$, so $B_a^{int} \leadsto C$ by Lemma 20. Hence, $A \leadsto B_a^{int} \leadsto C$. All of A, B_a^{int} and C have size at least N_2, so, by Lemma 24, there exist $A_1 \subseteq A$ and $C_1 \subseteq C$ such that $|A_1| \geqslant |A| - g$, $|C_1| \geqslant |C| - g$ and $C_1 \leadsto A_1$.

Claim 26.2.

$$|B'| \geqslant |B_a^{ext}| - 5g \geqslant d - 5g \geqslant N_2 + g.$$

Proof. By Lemma 20, $B_a \rightsquigarrow C_1$. Hence we have $B_a \rightsquigarrow C_1 \rightsquigarrow A_1$ and $|B_a|, |C_1|, |A_1| \geqslant N_2 + g$. Applying first Lemma 24 and then Lemma 21, we obtain the existence of $A_2 \subseteq A_1$ and $B_a^1 \subseteq B_a$ of respective size at least $|A_1| - 2g$ and $|B_a| - 2g$ such that $A_2 \rightsquigarrow_t B_a^1$.

Now, we have $A_2 \rightsquigarrow_t B_a^1$, $|A_2| \geqslant |A| - 3g \geqslant \frac{N_2}{t}$ and $|B_a^1| \geqslant |B_a| - 2g \geqslant 2d - 2g > \frac{N_2}{t}$. The maximality of $|A| + |B|$ ensures:

$$\begin{aligned}
|A| + |B| &\geqslant |A_2| + |B_a^1| \\
|A| + |B_a^{int}| + |B'| &\geqslant |A| - 3g + |B_a^{int}| + |B_a^{ext}| - 2g \\
|B'| \geqslant |B_a^{ext}| - 5g &\geqslant d - 5g
\end{aligned}$$

$\Diamond$

We shall now prove the existence of $A^* \subseteq A$ and $B^* \subseteq B' \cup B_a^{int} \cup B_a^{ext}$ each of size at least N_2/t such that $A^* \rightsquigarrow_t B^*$ and $|A^*| + |B^*| \geqslant |A| + |B|$, which contradicts the maximality of $|A| + |B|$. The proof is organized as follows: first using Lemmas 20, 21 (several times), 22 and 24, we show that almost all vertices of B' have many out-neighbours in C. Then we show the same for almost all vertices in B_a. Using this degree assumption and Lemma 21, we establish the existence of large sets $A_3 \subseteq A$ and $B_3 \subseteq B' \cup B_a$ such that $A_3 \rightsquigarrow B_3$. This main fact, combined with few other calculations lead to the existence of the above-mentioned sets A^* and B^*.

Since $A_1 \rightsquigarrow B$ and $B' \subseteq B$, we have $A_1 \rightsquigarrow B'$. Thus $C_1 \rightsquigarrow A_1 \rightsquigarrow B'$ and $|C_1|, |A_1|, |B'| \geqslant N_2$. So Lemma 24 ensures that there exist $C_2 \subseteq C_1$ and $B'_1 \subseteq B'$ such that $|C_2| \geqslant |C_1| - g$, $|B'_1| \geqslant |B'| - g$ and $B'_1 \rightsquigarrow C_2$. Now, since $B'_1 \rightsquigarrow C_2$ and $|B'_1|, |C_2| \geqslant N_2$, by Lemma 21, there exist $B'_2 \subseteq B'_1$ and $C_3 \subseteq C_2$ such that $|B'_2| \geqslant |B'_1| - g$, $|C_3| \geqslant |C_2| - g$, and $B'_2 \rightsquigarrow_t C_3$. So, for all $b \in B'_2$, we have:

$$d^+_{C_1}(b) \geqslant t \cdot |C_3| \geqslant t \cdot (|C_1| - 2g) = \left(t - \frac{2tg}{|C_1|}\right) \cdot |C_1|. \tag{2}$$

Lemma 20 ensures that $B_a \rightsquigarrow C_1$. Moreover $|B_a|, |C_1| \geqslant N_2$, so by Lemma 21, there exist $B_a^2 \subseteq B_a$ and $C_4 \subseteq C_1$ such that $|B_a^2| \geqslant |B_a| - g$, $|C_4| \geqslant |C_1| - g$ and $B_a^2 \rightsquigarrow_t C_4$. So, for all $b \in B_a^2$, we have:

$$d^+_{C_1}(b) \geqslant t|C_4| \geqslant t(|C_1| - g) = \left(t - \frac{tg}{|C_1|}\right) |C_1|. \tag{3}$$

Since $C_1 \rightsquigarrow A_1$ and $|C_1|, |A_1| \geqslant N_2$, Lemma 21 ensures the existence of $A_3 \subseteq A_1$ and $C_5 \subseteq C_1$ such that $|A_3| \geqslant |A_1| - g$, $|C_5| \geqslant |C_1| - g$ and $C_5 \rightsquigarrow_t A_3$. So, for all $a \in A_3$, we have:

$$d^-_{C_1}(a) \geqslant t|C_5| \geqslant t(|C_1 - g) \geqslant \left(t - \frac{tg}{|C_1|}\right) |C_1|. \tag{4}$$

Set $p = (2t - \frac{tg}{|C_1|})$. Then (2), (3) and (4) ensures that for all $b \in B'_2 \cup B_a^2$, $d^+_{C_1}(b) \geqslant p|C_1|$ and for all $a \in A_3$, $d^-_{C_1}(a) \geqslant p|C_1|$. Moreover, since $|C_1| > d - g \geqslant \frac{2tg}{(t-s)}$, we have $p > s$. Thus Lemma 22 yields

$$A_3 \rightsquigarrow B'_2 \cup B_a^2.$$

Let us apply Lemma 21 one last time. Indeed, both A_3 and $B'_2 \cup B^2_a$ have size at least N_1. Thus, there exist $A^* \subseteq A_3$ and $B^* \subseteq B'_2 \cup B^2_a$ of size respectively at least $|A_3| - g$ and $|B'_2 \cup B^2_a| - g$ such that $A^* \rightsquigarrow_t B^*$.

Observe that $|A^*| \geqslant |A_3| - g = |A_1| - 2g = |A| - 3g \geqslant \frac{N_2}{t}$. and $|B'_2| = |B'| - 2g$. Moreover $|B^*| \geqslant |B'_2| - g \geqslant |B'_1| - 2g \geqslant |B'| - 3g \geqslant d - 8g$ by Claim 26.2. Since $d \geqslant \frac{N_2}{t} + 8g$ by definition, we have $|B^*| \geqslant \frac{N_2}{t}$.

Furthermore the following inequalities are satisfied:

$$\begin{aligned}
|A^*| + |B^*| &\geqslant |A_3| + |B'_2| + |B^2_a| - 2g \\
&\geqslant |A| + |B^{int}_a| + |B^{ext}_a| + |B'| - 7g \\
&\geqslant |A| + |B| + |B^{ext}_a| - 7g \\
&> |A| + |B|.
\end{aligned}$$

The first inequality is due to the last extraction. The second comes from $|A_3| \geqslant |A_1| - g \geqslant |A| - 2g$ and $|B^2_a| \geqslant |B_a| - g = |B^{int}_a| + |B^{ext}_a| - g$ and $|B'_2| \geqslant |B'| - 2g$. Finally the last inequality comes from the fact that B^{ext} has size at least d which is greater than $7g$ by definition.

Thus $A^* \rightsquigarrow_t B^*$, $|A^*| + |B^*| > |A| + |B|$ and both A^* and B^* have size at least $\frac{N_2}{t}$, a contradiction to the maximality of $A \rightsquigarrow_t B$.

Case 2: $|B| \geqslant \frac{N_2}{t} + 4g$, and there exists $b \in B$ such that $d^+_{D-(A\cup B)}(b) \geqslant d$ and $d^-_{D-(A\cup B)}(b) \geqslant d$.

This case is analogous to Case 2 by directional duality.

Case 3: The remaining case.

Claim 26.1 ensures that there is a vertex x in $A \cup B$ with large in- and out-degree. Assume without loss of generality that $x \in A$. Since $|A| < \frac{N_2}{t} +$ by Case 2 and $|A| + |B| \geqslant 4d - 2g$ by Equation (1), we have $|B| \geqslant \frac{N_2}{t} + 4g$. So Case 2 ensures that no vertex b of B has in and out-degree at least d in the complement of $A \cup B$.

Let $b \in B$. Thus b has in-degree at most $d + \frac{N_2}{t} + 4g - 1$ (b can be incident to the vertices of A plus less than d vertices in $V \setminus (A \cup B)$) or b has out-degree at most d (there is no arc from B to A). But $d + \frac{N_2}{t} + 4g - 1 \leqslant 2d - 1$, which contradicts Lemma 25. $\square$

4 Forbidding Oriented Paths

4.1 Forbidding $P^+(3)$ or $P^+(1,1,1)$

Kierstead and Trotter [21] proved that Forb($P^+(3)$) is not χ-bounded. In fact, they show that an analogue of Zykov's construction of triangle-free graphs with arbitrarily large chromatic number yields acyclic $(TT_3, P^+(3))$-free oriented graphs with arbitrary large chromatic number. Interestingly, a result of Galeana-Sánchez et al. [11] implies that $\chi(\text{Forb}(\vec{C}_3, TT_3, P^+(3)) \cap \mathcal{S}) = 2$. Galeana-Sánchez et al. [11] studied 3-*quasi-transitive digraphs*, which are digraphs in which for every directed walk (u, v, w, z) either u and z are adjacent or $u = z$. In particular, every $(\vec{C}_3, TT_3, P^+(3))$-free oriented graph is 3-transitive. They characterized the strong 3-quasi-transitive digraphs. They showed that every such graph is either semicomplete, or semicomplete bipartite, or in the set $\mathcal{F}$ of oriented graphs D that have three vertices $\{v_1,v_2,v_3\}$ such that $A(D) = \{v_1v_2, v_2v_3, v_3v_1\} \cup \bigcup_{u\in V(D)\setminus\{v_1,v_2,v_3\}}\{v_1u, uv_2\}$. Recall that a digraph D is *semicomplete* if for any two vertices $u,v \in V(D)$ at least one of the two arcs uv and vu is in $A(D)$, and that it is *semicomplete bipartite* if there is a bipartition (A,B) of $V(D)$ such that if for any $a \in A$ and $b \in B$, at least one of the two arcs ab and ba is in $A(D)$. Since semicomplete digraphs and members of $\mathcal{F}$ are not $(\vec{C}_3, TT_3)$-free, strong $(\vec{C}_3, TT_3, P^+(3))$-free oriented graphs are bipartite tournaments and consequently have chromatic number at most 2. On the other hand, $\mathrm{Forb}(P^+(3)) \cap \mathcal{S}$ is not χ-bounded. Indeed, adding to every acyclic $(TT_3, P^+(3))$-free oriented graph D a vertex x which dominates all sources of D and is dominated by all other vertices, we obtain a strong $(\mathrm{Or}(K_4), P^+(3))$-free oriented graph D' with chromatic number $\chi(D)+1$; since $\chi(\mathrm{Forb}(\vec{C}, TT_3, P^+(3))) = +\infty$, we get $\chi(\mathrm{Forb}(\mathrm{Or}(K_4), P^+(3)) \cap \mathcal{S}) = +\infty$.

The *shift graph* $Sh_k(n)$, introduced by Erdős and Hajnal [9], is the graph whose vertices are the k-element subsets of $\{1,\dots,n\}$ and two vertices $a = \{a_1,\dots,a_k\}$ and $b = \{b_1,\dots,b_k\}$ are adjacent iff $a_1 < a_2 = b_2 < a_3 = b_3 < \cdots < a_{k-1} = b_{k-1} < b_k$. Gyárfás pointed out that the natural orientations of shift graphs are in $\mathrm{Forb}(\vec{C}_3, TT_3, P^+(1,1,1))$ but may have arbitrarily large chromatic number. Consequently, $\mathrm{Forb}(P^+(1,1,1))$ is not χ-bounded. Another way of seeing this is to note that every line oriented graph (i.e. an oriented graph which is a line digraph) is both TT_3-free and $P^+(1,1,1)$-free and that the line oriented graph of an acyclic oriented graph is also acyclic. Now, since it is well known that the chromatic number of the line digraph of D is at least $\log(\chi(D))$, this implies that the line oriented graphs of TT_n form a family of oriented graphs in $\mathrm{Forb}(\vec{C}_3, TT_3, P^+(1,1,1))$ with arbitrarily large chromatic number (which is consistent with Gyárfás's remark since natural orientations of shift graphs are in fact line oriented graphs). It can be deduced from Corollary 4.5.2 in [2] that in fact the class of line oriented graphs is exactly $\mathrm{Forb}(TT_3, P^+(1,1,1), C(3,1), C(2,2))$, where $C(3,1)$ (resp. $C(2,2)$) is the oriented cycle (a_1,a_2,a_3,a_4,a_1) such that $a_1 \to a_2 \to a_3 \to a_4 \leftarrow a_1$ (resp. $a_1 \to a_2 \to a_3 \leftarrow a_4 \leftarrow a_1$). It implies that $\mathrm{Forb}(\vec{C}, TT_3, P+(1,1,1), C(3,1), C(2,2))$ has unbounded chromatic number.

4.2 Forbidding $P^+(2,1)$

Theorem 27. $\chi(\mathrm{Forb}(\vec{C}_3, TT_3, P^+(2,1)) = 3$.

This result will be a consequence of the following lemma.

Lemma 28. *Let D be a $(\vec{C}_3, TT_3, P^+(2,1))$-free oriented graph.*

(1) Every oriented odd hole in D is directed.

(2) If a strong component of D contains an odd hole, then it is an initial strong component.

(3) If D is strongly connected, then there is a stable set that intersects every odd hole of D.

We first observe that this lemma implies Theorem 27.

Proof of Theorem 27 assuming Lemma 28. Let $D_1, \dots, D_p$ be the initial strong components of D. By (3), for every $1 \leqslant k \leqslant p$, there exists a stable set $S_k \subseteq V(D_k)$ such that $D_k - S_k$ has no odd holes. Now $S = S_1 \cup \cdots \cup S_p$ is also a stable set because there is no arc between two initial strong components, and by (1) and (2), S is a stable set that intersects every odd hole of D. Since D is $(\vec{C}_3, TT_3)$-free, this implies that $D - S$ is bipartite, which concludes the proof. $\square$

It remains to prove Lemma 28.

Proof of Lemma 28. (1) Every oriented odd hole contains a directed path of size at least 2. Thus unless it is directed it contains a $P^+(2,1)$.

Let us prove a claim that will imply (2) and (3). A vertex $x \in V(D) \setminus V(C)$ is a *C-twin of v_i* if $N^-(x) \cap V(C) = \{v_{i-1}\}$ and $N^+(x) \cap V(C) = \{v_{i+1}\}$ (indices are taken modulo q).

Claim 28.1. *Let $C = (v_1, \dots, v_q, v_1)$ be a directed odd cycle in D, and let x be a vertex in $V(D) \setminus V(C)$. Then:*

(i) x is dominated by at most one vertex of C.

(ii) If there is i such that x dominates v_{i+1} , then x is a C-twin of v_i.

(iii) If $x \in \mathrm{Reach}^-(C)$, then x is the C-twin of some v_i.

Proof. (i) Assume for a contradiction that x is dominated by two vertices in C. Without loss of generality, we may assume that these two vertices are v_1 and v_i with $i < q/2$. Then (v_q, v_1, x, v_i) is an induced $P^+(2,1)$ in D, a contradiction.

(ii) Assume that $x \to v_{i+1}$. The path $(v_{i-1}, v_i, v_{i+1}, x)$ is a $P^+(2,1)$. It is not induced, so $v_{i-1} \in N(x)$. If $x \to v_{i-1}$, then with the same reasoning $v_{i-3} \in N(x)$. We can repeat this process as long as $x \to v_{i+1-2j}$. However, this process has to stop since x is not adjacent to $v_{i+2} = v_{i+1-2\lfloor q/2 \rfloor}$. Consequently, there exists j such that $x \leftarrow v_{i+1-2j}$ and $x \to v_{i+1-2j'}$ for all $0 \leqslant j' < j$. But $j = 1$ for otherwise $(v_{i+1-2j}, x, v_{i+1}, v_i)$ is an induced $P^+(2,1)$. Hence, $v_{i-1} \to x$.

Now x does not dominate any vertex $v_j \in V(C) \setminus \{v_{i+1}\}$ for otherwise by the above reasoning both v_{j-2} and v_{i-1} would dominate x, a contradiction to (i). Therefore x is a C-twin of v_i.

(iii) Assume for a contradiction that $x \in \mathrm{Reach}^-(C)$ and x is not the C-twin of any v_i. Let P be a shortest dipath from x to C. Such a dipath exists because $x \in \mathrm{Reach}^-(C)$, and by (ii), P has length at least 2. Let v_{i+1} be the terminal vertex of P, u its in-neighbour in P and t the in-neighbour of u in P. The path (t, u, v_{i+1}, v_i) is a $P^+(2,1)$, which is not induced, so t and v_i are adjacent. But t does not dominate v_i since P is a shortest dipath from x to C, so $v_i \to t$.

Since u dominates v_{i+1}, we obtain that u is a C-twin of v_i by (ii). Therefore $C' = (v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_q, v_1)$ is also a directed odd cycle. By (ii), t is a C'-twin of v_{i-1}. In particular, $v_{i-2} \to t$. This gives a contradiction to (i) as t is dominated by v_{i-2} and v_i. $\Diamond$

(2) It now clearly follows from Claim 28.1 (iii).

(3) Suppose that D is strongly connected. If D contains no oriented odd cycle, then the result holds with $S = \emptyset$. If D contains an odd cycle $C = (v_1, \dots, v_q, v_1)$, then it is directed by (1) and by Claim 28.1, every vertex of D is the C-twin of some v_i. For $1 \leqslant i \leqslant q$, let T_i be the set C-twins of v_i plus v_i. Observe that if $xy \in A(D)$ with $x \in T_i$ and $y \in T_j$, then $|i - j| = 1 \mod q$, for otherwise (v_{i-1}, x, y, v_{j-1}) would be an induced $P^+(2,1)$. It follows that $D - T_1$ has no odd cycles, and T_1 is a stable set because all vertices in T_1 are in $N^-(v_2)$. Thus T_1 is our desired S. □

Remark 29. Wang and Wang [31] studied a class of digraphs that contains $\text{Forb}(\vec{C}_3, TT_3, P^+(2,1))$. A digraph is *arc-locally in-semicomplete* if for any pair of adjacent vertices x, y, every in-neighbour of x and every in-neighbour of y are either adjacent or the same vertex. Observe that the oriented graphs of $\text{Forb}(\vec{C}_3, TT_3, P^+(2,1))$ are arc-locally in-semicomplete. In particular, [31] characterizes strong arc-locally in-semicomplete digraphs. This characterization implies that every strong oriented graph in $\text{Forb}(P^+(2,1))$ is either a bipartite tournament (i.e. the orientation of a complete bipartite graph) or an extension of a directed cycle. This directly implies Lemma 28 (3).

4.2.1 Forbidding TT_3 and $P^+(2,1)$.

We shall now prove that $\chi(\text{Forb}(TT_3, P^+(2,1))) = 4$ and that this result is tight. Here is a short sketch of the proof. We first describe precisely the structure of a strong $(TT_3, P^+(2,1))$-free oriented graph that contains an odd hole (see Lemma 30). This permits us to colour such oriented graphs, more precisely we distinguish between two cases, if the oriented graph contains an odd hole of length 7 or more, then it is 3-colourable; if it contains an odd hole of length 5, then it is 4-colourable. We also give a tight example in the second case (see Lemmas 31 and 32). Finally we show how to 4-colour any $(TT_3, P^+(2,1))$-free oriented graph (Theorem 33).

Lemma 30. *Let D be a digraph in* $\text{Forb}(TT_3, P^+(2,1))$, *and let* $H = (v_1, \dots, v_{2k+1}, v_1)$, $k \geqslant 2$, *be an odd hole in D. Then:*

(i) H is directed.

(ii) If $u \in \text{Reach}^-(H) \setminus V(H)$, then u is adjacent to some vertex of H.

(iii) If v dominates a vertex in $V(H)$, then either there is an index i such that $vv_i, v_{i-2}v$ are the only two arcs between v and $V(H)$ or $k = 2$ and there are exactly three arcs between $V(H)$ and v and these are either $vv_i, v_{i-2}v, vv_{i+2}$ or $vv_i, v_{i-2}v, v_{i+1}v$ for some $i \in \{1, \dots, 5\}$.

(iv) If $N^+(v) \cap V(H) = \emptyset$ but $N^-(v) \cap V(H) \neq \emptyset$, then $|N^-(v) \cap V(H)| = 1$.

(v) H is contained in an initial strong component of D.

Proof. In all this proof, indices of the v_i are modulo $2k+1$.

(i) Every oriented odd hole contains a directed path of size at least 2. Thus, unless it is directed, it contains a $P^+(2,1)$.

(ii) Let u be a vertex in $\text{Reach}^-(H) \setminus V(H)$. Let $P = (x_0, x_1 \dots, x_q)$ be a shortest $(u, V(H))$-dipath. (Hence $u = x_0$). If $q = 1$ there is nothing to prove, so assume $q \geqslant 2$. We may assume, by relabelling $V(H)$ if necessary, that $x_q = v_{2k+1}$. As D is TT_3-free, the vertices x_{q-1} and v_{2k} are not adjacent. Consequently, as D is $P^+(2,1)$-free, x_{q-2} must be adjacent either to v_{2k+1} or to v_{2k}. By the minimality of P, the arc will enter x_{q-2} in both cases. Thus, since D is TT_3-free, D contains exactly one of those arcs. If $q = 2$, we are done since u is adjacent to a vertex of H. So suppose $q \geqslant 3$. If $v_{2k} \to x_{q-2}$ (resp. $v_{2k+1} \to x_{q-2}$), then since D is $(TT_3, P^+(2,1))$-free, the vertices v_{2k-1} (resp. v_{2k}) and x_{q-3} are adjacent, so, by minimality of P, $v_{2k-1} \to x_{q-3}$ (resp. $v_{2k} \to x_{q-3}$). And so on by induction, one proves that there is an arc from H to x_{q-4}, x_{q-5}, until we get an arc from H to u. This proves (ii).

(iii) Let v be a vertex in $V(D) \setminus V(H)$ that dominates a vertex, say v_i, in H. Moreover, without loss of generality, we may assume that vv_{i-2} is not an arc. Indeed if v dominates v_{i-2} for all i, then v would dominate all vertices of H and D would contain a TT_3.

Since D is TT_3-free, then v and v_{i-1} are not adjacent. Now there can be no arc $v_j v$ with $j \notin \{i-2, i, i+1\}$ for otherwise (v_j, v, v_i, v_{i-1}) would be an induced $P^+(2,1)$. Furthermore, since D has no induced $P^+(2,1)$, there is an arc between v and v_{i-2}. By our assumption, this arc is $v_{i-2}v$. Now, there can be no arc vv_j with $j \notin \{i-3, i\}$ for otherwise $(v_{i-2}, v, v_j, v_{j-1})$ would be an induced $P^+(2,1)$ or $D\langle\{v, v_{j-1}, v_j\}\rangle$ would be a TT_3. Consequently, in addition to vv_i and $v_{i-2}v_i$, the only possible arcs between v and H are $v_{i+1}v$ and vv_{i-3}. If $k \geqslant 3$, then $vv_{i-3} \notin A(D)$ for otherwise $(v_{i-5}, v_{i-4}, v_{i-3}, v)$ is an induced $P^+(2,1)$, and $v_{i+1}v \notin A(D)$, for otherwise $(v_{i-3}, v_{i-2}, v, v_{i+1})$ is an induced $P^+(2,1)$.

If $k = 2$, then $i-3 = i+2$. Both $v_{i+1}v$ and vv_{i+2}, cannot be arcs for otherwise $\{v, v_{i+1}, v_{i+2}\}$ induces a TT_3. This completes the proof of (iii).

(iv) Assume for a contradiction that $N^+(v) \cap V(H) = \emptyset$ and $|N^-(v) \cap V(H)| \geqslant 2$. There are distinct indices i and j such that $v_i v$ and $v_j v$ are arcs. Observe that $i \notin \{j-1, j+1\}$ because D has no TT_3, and v_{j-1} and v are not adjacent because $N^+(v) \cap V(H) = \emptyset$ and D has no TT_3. If $|j-2| \neq 2$ then (v_{j-1}, v_j, v, v_i) is an induced $P^+(2,1)$, and if $i = j-2$ then (v_{j-1}, v_i, v, v_j) is an induced $P^+(2,1)$. In both cases, this is a contradiction.

(v) Suppose for a contradiction that H is contained in a strong component C that is not initial. Then there is a vertex $u \in \mathrm{Reach}^-(H) \setminus V(C)$ that belongs to an initial component. By (ii), u is adjacent to a vertex in H. If u dominates a vertex in H, then by (iii) it is also dominated by a vertex of H. Hence in any case, u is dominated by a vertex of H. But this implies that $u \in C$, a contradiction. $\square$

Lemma 31. *Let D be a strong digraph in* $\mathrm{Forb}(TT_3, P^+(2,1))$. *If D contains an odd hole H with at least 7 vertices, then D is an extension of H. In particular $\chi(D) = 3$.*

Proof. Let $H = (v_1, \dots, v_{2k+1}, v_1)$, $k \geqslant 3$ be an odd hole in D. By Lemma 30 (i)–(ii), H is directed and every vertex of $V(D) \setminus V(H)$ is adjacent to $V(H)$. Suppose D is not an extension of H. Then by Lemma 30 (iii)–(iv) there is a vertex x_1 such that $N^+(x_1) \cap V(H) = \emptyset$ and $|N^-(x_1) \cap V(H)| = 1$. Let v_j be the vertex of $N^-(v) \cap V(H)$. As D is strong there exists a (x_1, H)-dipath. Let $P = (x_1, x_2, \dots, x_t, v_i)$ be a shortest such dipath. Then by minimality of P, v_t is the only vertex of $P - \{v_i\}$ that has an arc to $V(H)$. By Lemma 30 (iii), $v_{i-2}w, wv_i$ are the only arcs between w and $V(H)$. Now, x_{t-1} must be adjacent to v_{i-3}, for otherwise $(v_{i-3}, v_{i-2}, x_t, x_{t-1})$ is an induced $P^+(2,1)$. As x_{t-1} has no arc to $V(H)$ we have that $v_{i-3}x_{t-1}$ is an arc and by

Lemma 30 (iv) this is the only arc between x_{t-1} and $V(H)$, implying that $(x_{t-1}, x_t, v_i, v_{i-1})$ is an induced $P^+(2,1)$, a contradiction. □

Lemma 32. *Let D be a strong $(TT_3, P^+(2,1))$-free oriented graph. If D contains a 5-hole, then $\chi(D) \leqslant 4$.*

Proof. Let $H = (v_1, v_2, v_3, v_4, v_5, v_1)$ be a 5-hole in D. For $i = 1, \dots, 5$, define (subscripts are taken modulo 5 all along the proof):

- $A_i = \{v \in V(D) \setminus V(H) : v \leftarrow v_{i-1}$, and $v \rightarrow \{v_{i+1}, v_{i+3}\}$.
- $B_i = \{v \in V(D) \setminus V(H) : v \leftarrow \{v_{i-1}, v_{i+2}\}$, and $v \rightarrow v_{i+1}\}$.
- $C_i = \{v \in V(D) \setminus V(H) : v \rightarrow v_{i-1}$ and $v \leftarrow v_{i+1}\ \}$.
- $X_i = A_i \cup B_i \cup C_i$.

Similarly to the proof of Lemma 31, one shows that there is no vertex x_1 such that $N^+(x_1) \cap V(H) = \emptyset$ and $|N^-(x_1) \cap V(H)| = 1$. Thus, by Lemma 30 (ii)-(iv), the sets $X_1, \dots, X_5$ partition the set $V(D) \setminus V(H)$. Moreover, since D is TT_3-free, we have:

Claim 32.1. *For $i = 1, \dots, 5$, X_i is a stable set, and there is no arc between X_i and B_{i+2} or between X_i and A_{i+3}.*

Let π be the colouring of D defined as follows (see Figure 3).

- $\pi(v_1) = 1$, $\pi(v_2) = \pi(v_5) = 2$, $\pi(v_3) = 3$ and $\pi(v_4) = 4$;
- $\pi(x) = 1$ for all $x \in X_1 \cup A_4$;
- $\pi(x) = 2$ for all $x \in X_2 \cup B_2 \cup A_5 \cup C_5$;
- $\pi(x) = 3$ for all $x \in X_3 \cup B_5$;
- $\pi(x) = 4$ for all $x \in A_2 \cup B_4 \cup C_4$.

By Claim 32.1, π is a proper colouring of $D - C_2$.

For any $v \in C_2$, set $\pi(v) = 4$ if v has a neighbour in C_5, and $\pi(v) = 2$ otherwise. We shall prove that the function π is a proper colouring of D. By Claim 32.1, if $v \in C_2$ has no neighbour in C_5, then none of its neighbours is coloured 2. So the only problem that might occur is if a vertex of $v \in C_2$ coloured with 4 (and thus adjacent to a vertex $u \in C_5$) has a neighbour with colour 4, say w. By Claim 32.1, $w \in C_4$ and since D is TT_3-free, vu and wv are arcs of D.

If u and v were not adjacent, then (w, v, u, v_4) would be an induced $P^+(2,1)$. So they are adjacent and u dominates w, since (u, vw) cannot induce a TT_3. But then (v_2, v_3, w, u) is an induced $P^+(2,1)$, a contradiction. This proves that π is a proper colouring of D and then $\chi(D) \leqslant 4$. □

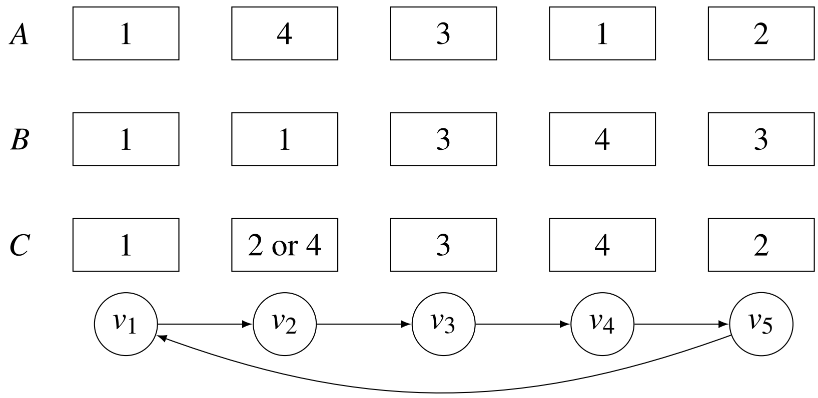

Figure 3: The colouring π of $D - C_2$

We now describe a $(TT_3, P^+(2,1))$-free oriented graph with chromatic number 4. We take two 5-holes $C_1 = (v_1, v_2, v_3, v_4, v_5, v_1)$ and $C_2 = (u_1, u_2, u_3, u_4, u_5, u_1)$ and for each vertex u_i we add the arcs $v_{i-1}u_i, u_iv_{i+1}$ and u_iv_{i+3} (see Figure 4). It is a routine exercise to check that this oriented graph is indeed $(TT_3, P^+(2,1))$-free. In any 3-colouring of C_1, there exists $i \in \{1, \dots, 5\}$ such that the vertices $v_{i-1}, v_{i+1}, v_{i+3}$ have distinct colours, and thus no colour is available for u_i. So this graph is not 3-colourable.

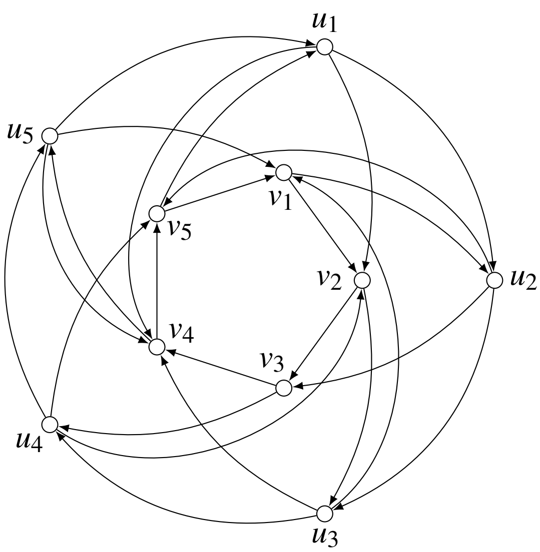

Figure 4: A $(TT_3, P^+(2,1))$-free oriented graph with chromatic number 4.

Theorem 33. $\chi(\mathrm{Forb}(TT_3, P^+(2,1))) = 4$. *More precisely, we prove that if D is a $(TT_3, P^+(2,1))$-free oriented graph, then the following hold.*

- $\chi(D) \leqslant 4$;
- *If D contains an odd hole of length 7 or more, then $\chi(D) = 3$.*

Proof. Let $D \in \text{Forb}(TT_3, P^+(2,1))$ and assume D is connected. We may assume that D admits at least one initial strong component K that contains an odd hole, otherwise by Lemma 30 (v) D is odd hole-free and thus is 4-colourable by Theorem 3.

Claim 33.1. *K is the only initial strong component of D.*

Proof. Assume D contains another initial strong component K'. Let $P = (p_1, p_2, \dots, p_k)$ be a shortest path from K to K', where $p_1 \in K$ and $p_k \in K'$. Note that since K and K' are initial strong components $p_1 \rightarrow p_2$ and $p_k \rightarrow p_{k-1}$. There exists a vertex p_0 in $V(K) \setminus \{p_1\}$ such that $p_0 \rightarrow p_1$. Observe that by minimality of P, and since D is TT_3-free, $P' = (p_0, p_1, \dots, p_k)$ is an induced path. Moreover, since $p_0 \rightarrow p_1 \rightarrow p_2$ and $p_{k-1} \leftarrow p_k$, necessarily P' contains a $P^+(2,1)$, a contradiction. $\Diamond$

Let $\text{dist}(K,x)$ denote the distance from K to x, that is the length of a shortest dipath from k to x in D. Note that $\text{dist}(K,x)$ is well-defined for every vertex $x \in V(D)$, because K is the only initial strong component, so every vertex can be reached from K. Set $L_i = \{x : \text{dist}(K,x) = i\}$ (in particular $L_0 = K$). Clearly, the L_i partition $V(D)$. If $j > i$, an arc from L_j to L_i is called a *backward arc*.

Claim 33.2. *D has no backward arcs.*

Proof. Assume for contradiction that uv is a backward arc from L_j to L_i and assume it has been chosen with respect to the minimality of i. Observe that $i \geqslant 1$. If $i \geqslant 2$, then there exists a vertex $v_1 \in L_{i-1}$ and a vertex $v_2 \in L_{i-2}$ such that $v_2 \rightarrow v_1 \rightarrow v$ and thus (v_2, v_1, v, u) is a $P^+(2,1)$ and it is induced by minimality of i, a contradiction. So we may assume that $i = 1$. Let $v_1 \in V_0$ such that $v_1 \rightarrow v$. There exists a vertex $v_2 \in L_0$ such that $v_2 \rightarrow v_1$ and since D is TT_3-free, v_2 is not adjacent to v. Hence $\{v_2, v_1, v, u\}$ induces a $P^+(2,1)$, a contradiction. $\Diamond$

Claim 33.3. *For any $i \geqslant 2$, L_i is a stable set.*

Proof. Let $i \geqslant 2$ and assume that uv is an arc of L_i. There exists $v_1 \in L_{i-1}$ and $v_2 \in L_{i-2}$ such that $v_2 \rightarrow v_1 \rightarrow v$. So (v_2, v_1, v, u) is a $P^+(2,1)$ and it is induced since there is no TT_3 nor backward arcs. $\Diamond$

A *directed bipartite graph* is an orientation of a connected bipartite graph such that every vertex is either a source or a sink.

Claim 33.4. *L_1 is a disjoint union of directed bipartite graphs.*

Proof. Assume for contradiction that there exists $a, b, c \in L_1$ such that $a \rightarrow b \rightarrow c$ (note that ca might or might not be an arc). We distinguish between two cases.

Case 1: c admits a neighbour $c_1 \in L_0$ such that c_1 belongs to an odd hole $H = (c_1, \dots, c_{2k+1}, c_1)$ of L_0. Since (c_{2k+1}, c_1, c, b) cannot be induced, $c_{2k+1} \rightarrow b$ and since (c_{2k}, c_{2k+1}, b, a) cannot be induced, $c_{2k} \rightarrow a$. Recall that by Lemma 30 (iv), a vertex in L_1 is adjacent to at most one vertex in H. Since (c_1, c, b, a) cannot be induced, $c \rightarrow a$. But now (c_1, c, a, c_{2k}) is an induced $P^+(2,1)$, a contradiction.

Case 2: no neighbour of c in L_0 belongs to an odd hole in L_0. Let $c_1 \in L_0$ be a neighbour of c. By Lemma 31, if L_0 contains an odd hole of length at least 7, then all vertices of L_0 belong to an odd hole. So we may assume that L_0 contains a 5-hole, say $H = (u_1, u_2, u_3, u_4, u_5, u_1)$. By Lemma 30 (iii), we may assume without loss of generality that $u_2 \to c_1 \to u_4$ and that exactly one of u_5c_1, c_1u_1 is an arc. Recall again that by Lemma 30 (iv), a vertex in L_1 is adjacent to at most one vertex in H.

Since (u_2, c_1, c, b) cannot be induced, u_2b is an arc. Since (u_1, u_2, b, a) cannot be induced, u_1a is an arc. Since (a, b, c, c_1) cannot be induced and c_1a is not an arc by Lemma 30 (iv), c and a are adjacent and we have $c \to a$. But now (u_5, u_1, a, c) is an induced $P^+(2,1)$ (it is indeed induced because c has no neighbour in H), a contradiction. $\lozenge$

We may now assume that L_1 consists of t directed bipartite graphs $(A_1, B_1), \dots, (A_t, B_t)$ such that all arcs of L_1 are from A_i to B_i.

Claim 33.5. *Let $1 \leqslant i \leqslant t$ and let $u, v \in A_i$. Then u and v have the same neighbourhood in L_0 and the graph induced by $N_{L_0}(B_i)$ and $N_{L_0}(A_i)$ is a complete bipartite graph.*

Proof. Assume for a contradiction that there exists a vertex $u' \in L_0$ such that $u'u$ is an arc but $u'v$ is not. We may assume without loss of generality that u and v have a common neighbour in B_i, say w. Then $u'uwv$ induces a $P^+(2,1)$, a contradiction.

Let $w \in B_i$ and w' be a neighbour of w in L_0. Let $u \in A_i$ be a neighbour of w. Let $u' \in N_{L_0}(A_i)$. Since u' dominates all vertices of A_i, u' dominates u and thus $u' \neq w$, otherwise $\{u', u, w\}$ is a TT_3, and u' is adjacent to w', otherwise $\{u', u, w, w'\}$ induces a $P^+(2,1)$. $\lozenge$

Claim 33.6. *Let $i \geqslant 2$ and let $u \in L_i$. Then the neighbours of u in L_{i-1} have the same neighbourhood in L_{i-2}.*

Proof. Let v, w be two neighbours of u in L_{i-1}. Since there is no backward arcs, vu and wu are arcs. If some $z \in L_{i-2}$ was adjacent to precisely one of v, w, say v, then $\{z, v, u, w\}$ would induce a $P^+(2,1)$. Hence v and w share the same in-neighbourhood, which implies the claim. $\lozenge$

We are now going to explain how a k-colouring of L_0 (where $k = 3$ or 4), can be extended to the rest of the graph. So assume that L_0 is coloured with colours from $\{1, 2, \dots, k\}$.

We start by colouring L_1. Let $1 \leqslant i \leqslant t$ and let $I \subseteq \{1, \dots, k\}$ be the set of colours used to colour $N_{L_0}(A_i)$. Since $N_{L_0}(B_i)$ is complete to $N_{L_0}(A_i)$, $I \neq \{1, \dots, k\}$ and colours from $\{1, \dots, k\} - I$ are used to colour $N_{L_0}(B)$. So we can colour the vertices of A_i with a colour from $\{1, \dots, k\} - I$ and the vertices in B_i with a colour from I. Hence we can colour all vertices of L_1. Moreover assume we are doing so in such a way that two vertices of L_1 that are sharing the same neighbourhood in L_0 are coloured with the same colour.

Now we colour the rest of the graph layer by layer. Assume that all layers below L_i ($i \geqslant 2$) have already been coloured in such a way that two vertices in the same layer that have a common neighbour in the layer below are coloured with the same colour. Then, by Claim 33.6, each vertex in L_i sees a single colour in L_{i-1}, so it is easy to extend the colouring. $\square$

4.3 Forbidding several orientations of P_4

Observe that, by directional duality, $\chi(\text{Forb}(P^+(3), P^+(2,1))) = \chi(\text{Forb}(P^+(3), P^-(2,1)))$.

Proposition 34. *An oriented graph in* $\text{Forb}(P^+(3), P^+(2,1))$ *or* $\text{Forb}(P^+(3), P^+(1,1,1))$ *contains no odd hole.*

Proof. Let D be a $(P^+(3), P^+(2,1))$-free oriented graph. Assume for a contradiction, that it contains an odd hole $C = (v_1, \dots, v_p, v_1)$. Necessarily, C contains two consecutive edges that are oriented in the same direction. Without loss of generality, $v_1 \to v_2 \to v_3$. Now (v_1, v_2, v_3, v_4) is either a $P^+(3)$ or a $P^+(2,1)$, a contradiction.

Let D be a $(P^+(3), P^+(1,1,1))$-free oriented graph. Assume for a contradiction, that it contains an odd hole $C = (v_1, \dots, v_p, v_1)$. Necessarily, C contains two edges at distance 1 that are oriented in the same direction. Without loss of generality, $v_1 \to v_2$ and $v_3 \to v_4$. Now (v_1, v_2, v_3, v_4) is either a $P^+(3)$ or a $P^+(1,1,1)$, a contradiction. $\square$

A recent and difficult paper of Seymour and Scott (see [28]) proves that the class of odd-hole-free graphs is χ-bounded, which directly yields the following results.

Corollary 35. $\text{Forb}(P^+(3), P^+(2,1))$, $\text{Forb}(P^+(3), P^-(2,1))$, *and* $\text{Forb}(P^+(3), P^+(1,1,1))$ *are* χ*-bounded.*

A natural question is to ask for the values (or nice bounds) of $\chi(\text{Forb}(\text{Or}(K_k), P^+(3), P^+(2,1)))$ and $\chi(\text{Forb}(\text{Or}(K_k), P^+(3), P^+(1,1,1)))$ for every $k \geqslant 3$. A graph with no odd hole nor clique of size 3 contains no odd cycle and thus is bipartite. Thus

Proposition 36. $\chi(\text{Forb}(\vec{C}_3, TT_3, P^+(3), P^+(2,1))) = \chi(\text{Forb}(\vec{C}_3, TT_3, P^+(3), P^+(1,1,1))) = 2.$

One can also easily prove the following proposition.

Proposition 37.

$$\chi(\text{Forb}(\vec{C}_3, TT_3, P^+(2,1), P^+(1,1,1))) = 3.$$

This proposition also derives directly from Theorem 27 and the fact that directed odd cycles are in $\text{Forb}(\vec{C}_3, TT_3, P^+(2,1), P^+(1,1,1))$.

5 Concluding Remarks

Let us conclude by discussing the remaining open cases. Conjecture 4 about stars is still widely open with the next case to study being $\text{Forb}(\text{Or}(K_4), S_{k,k})$. About oriented paths, note that since $\text{Forb}(P^+(3))$ and $\text{Forb}(P^+(1,1,1))$ are not χ-bounded, the only open cases for orientations of P_k that would be χ-bounding are paths of the type $P^+(2,2,\dots,2)$ or $P^+(1,2,2,\dots,2)$, or $P^+(1,2,2,\dots,2,1)$ (following our notations). In fact for trees in general, most orientations will contain either $P^+(3)$ and $P^+(1,1,1)$ and hence when forbidden will define classes that are not χ-bounded.

Recall that Conjecture 1 states that for every tree T, the class of T-free graphs is χ-bounded. A stronger conjecture could be the following : for every tree T, there exists one orientation $\vec{T}$ of T such that the class of graphs that admit a $\vec{T}$-free orientation is χ-bounded. This is false for many trees, as shown below.

Proposition 38. *There exists a tree T such that for every orientation $\vec{T}$ of T, $\mathrm{Forb}(\vec{T})$ is not χ-bounded.*

Proof. To construct T, start with an induced path on four vertices $\{v_1, v_2, v_3, v_4\}$ and add vertices $\{w_1, w_2, w_3, w_4\}$ such that $N(w_i) = \{v_i\}$. It is easy to see that every orientation of this tree contains either a $P^+(3)$ or $P^+(1,1,1)$. Therefore $\mathrm{Forb}(\vec{T})$ contains either $\mathrm{Forb}(P^+(3))$ or $\mathrm{Forb}(P^+(1,1,1))$ which are both not χ-bounded. □

Of course any tree that contains this tree T will also satisfy the theorem. Up to our knowledge, Gyárfás-Sumner conjecture (Conjecture 1) is not known to be true for these trees, so they could be natural candidates for counterexamples.

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